The Shape of Planck's Law

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Abstract

Planck's Law describes the spectral density of electromagnetic radiation emitted by a black body at a temperature T. While it is well known that the "shape" of Planck's Law is independent of the temperature, it is not clear what is meant by the shape of a function. In this note, a notion of shape is introduced and Planck's Law is shown to have the same shape for every temperature. This property of a family of functions can be useful for computing properties of the functions.

Introduction

PLANCK'S LAW is a function that describes an important property of entities in thermal equilibrium. In this note, we introduce a simple notion of "shape" and show that Planck's Law has the same shape for any temperature. Planck's Law at a particular temperature is actually several functions, depending on the spectral variable, such as frequency, wavelength, wavenumber and the angular versions of these three. Each of these forms of Planck's Law is a family of functions, and the functions within one family have the same shape. We illustrate the shape of the forms of Planck's Law for frequency and wavelength.

The notion of the shape of a function can be useful for computing properties of families of functions, all of which have the same shape. We illustrate this by showing some properties of the forms of Planck's Law for frequency and wavelength.

The Shape of a Function

In geometry, two subsets of a Euclidean space have the *same shape* if one can be transformed to the other by a combination of translations, rotations (together also called rigid transformations), and uniform scalings (Kendall, 1984). Sometimes mirror images are also considered to have the same shape. While the graph of a function is a geometrical figure, this definition of shape is not entirely satisfactory, because the domain and range should be preserved by the transformation. Accordingly, we define two functions f and g to have the same shape if there is a

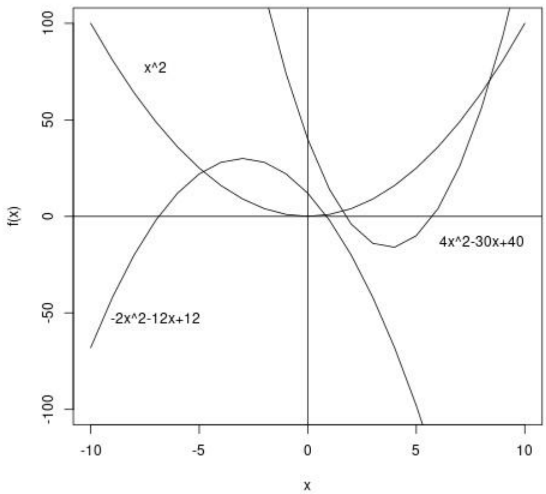

Figure 1: Some examples of quadratic functions

nonsingular affine transformation that bijectively maps the graph of f onto the graph of g, and that the affine transformation is the direct product of transformations on the domain and range, each of which is a combination of rigid transformations, uniform scalings and/or reflections. For the case of real-valued functions of a single variable, $f(x)$ and $g(x)$, this means that there are constants $p \neq 0$, q, $r \neq 0$, and s, such that for every x, $f(x) = pg(rx+s)+q$, and for every y, $g(y) = \frac{1}{p} f(\frac{y-s}{r}) - \frac{q}{p}$.

To understand the notion of the shape of a function, we first consider a simpler case: quadratic functions. If one plots some examples of quadratic functions as in Figure 1, it seems clear that all quadratic functions have the same shape. In general, if $f(x)$ be the quadratic function ax^2+bx+c, where $a \neq 0$, Then one can transform $f(x)$ to x^2 by using $p = a$, $q = \frac{4ac-b^2}{4a}$, $r = 1$, and $s = \frac{b}{2a}$. In other words, one can "normalize" every quadratic function to the "standard" quadratic function x^2. Thus every quadratic function has the same shape. For example, the quadratic function given by $f(x) = -2x^2 - 12x + 12$ in Figure 1, can be transformed to x^2 by $f(x) = -2(x+3)^2 + 30$.

The special case where the transformation does not include a translation is especially useful because the zeroes of the function and all of its derivatives transform in the same way as the function. More precisely, if $f(x) = pg(rx)$, then x_0 is a zero of f if and only if x_0/r is a zero of g. Since $f'(x) = prg'(rx)$, it follows that x_1 is a zero of f' if and only if x_1/r is a zero of g', and similarly for higher derivatives. In particular, the maxima, minima and inflection points of f all transform to maxima, minima and inflection points of g by the same transformation.

Planck's Laws

Planck's Law with respect to the frequency of electromagnetic radiation is the spectral emissive power per unit area, per unit solid angle, per unit frequency. The formula is

$$B_\nu(\nu,T) = \frac{2h\nu^3}{c^2}\left(e^{\frac{h\nu}{kT}} - 1\right)^{-1} \tag{1}$$

where ν is the frequency in Hz, T is the absolute temperature in kelvins, h is Planck's constant, c is the speed of light in a vacuum, and k is the Boltzmann constant. The SI units of B_ν are $W \cdot sr^{-1} \cdot m^{-2} \cdot Hz^{-1}$.

Another form of Planck's Law uses wavelength rather than frequency. The formula is

$$B_\lambda(\lambda,T) = \frac{2hc^2}{\lambda^5}(e^{\frac{hc}{kT\lambda}} - 1)^{-1} \tag{2}$$

where λ is the wavelength. This form is not the same as simply expressing $B_\nu(\nu,T)$ in terms of λ. The electromagnetic radiation is measured as the spectral emissive power per unit area, per unit solid angle, per unit wavelength. The SI units of B_λ are $W \cdot sr^{-1} \cdot m^{-3}$.

The Shape of Planck's Law

We now show that each of the forms of Planck's Law in Equations (1) and (2) have the property that all functions in each family have the same shape. Looking at Equation (1), it is not easy to see how one might transform the functions for different temperatures into each other. The way to do this is to "normalize" the frequency and spectral density to obtain a function that does not depend on the temperature T, as we did for quadratic functions.[1] In fact, it should be easier because we now have just one parameter T, while the family of quadratic functions has three parameters, a, b and c. First consider the family of functions $B_\nu(\nu,T)$ defined in Equation (1). We want to find $p \neq 0$ and $r \neq 0$ such that the function $pB_\nu(rx,T)$ does not depend on T. Note that both q and s must be 0 because the domain and range of B_ν are the positive real numbers. The parameter T only occurs once in $B_\nu(\nu,T)$, and setting $r = T$ will cancel that occurrence of T as follows:

$$B_\nu(Tx,T) = \frac{2h(Tx)^3}{c^2}\left(e^{\frac{h(Tx)}{kT}} - 1\right)^{-1} = \frac{2hT^3x^3}{c^2}\left(e^{\frac{h\nu}{k}} - 1\right)^{-1} \tag{3}$$

This eliminated the parameter T in the exponent, but now T occurs in another location. However, this is easily eliminated by setting $p = T^{-3}$ to get

$$T^{-3}B_\nu(Tx,T) = T^{-3}\frac{2hT^3x^3}{c^2}\left(e^{\frac{hx}{k}} - 1\right)^{-1} = \frac{2hx^3}{c^2}\left(e^{\frac{hx}{k}} - 1\right)^{-1} \tag{4}$$

This solves the problem of showing that the family of Planck's Law functions with respect to the frequency all have the same shape. It also proves that the maximum of any member of this family occurs at a frequency that is proportional to T. A similar process shows that the family of functions $B_\lambda(\lambda,T)$ have the same shape and that the maximum value occurs at a wavelength that is proportional to T^{-1}. The latter fact is known as Wien's Displacement Law.

A better normalization of B_ν would be to simplify the function as much as possible as follows:

$$\frac{h^2c^2}{2k^3T^3}B_\nu\left(\frac{kT}{h}x,T\right) = \frac{h^2c^2}{2k^3T^3}\frac{2h}{c^2}\left(\frac{kT}{h}x\right)^3(e^x - 1)^{-1} = \frac{x^3}{e^x - 1} \tag{5}$$

The normalized function in Equation (5) is shown in Figure 2, along with the first and second derivatives.

Similarly, one can simplify B_λ as much as possible as follows:

$$\frac{h^4c^3}{2k^5T^5}B_\lambda\left(\frac{hc}{kT}x,T\right) = \frac{h^4c^3}{2k^5T^5}\frac{2hc^2}{(\frac{hcx}{kT})^5}(e^{1/x} - 1)^{-1} = \frac{1}{(e^{1/x} - 1)x^5} \tag{6}$$

[1] Such a normalized function need not necessarily be a Planck's Law function for any temperature, but it is mathematically a function with the same shape.

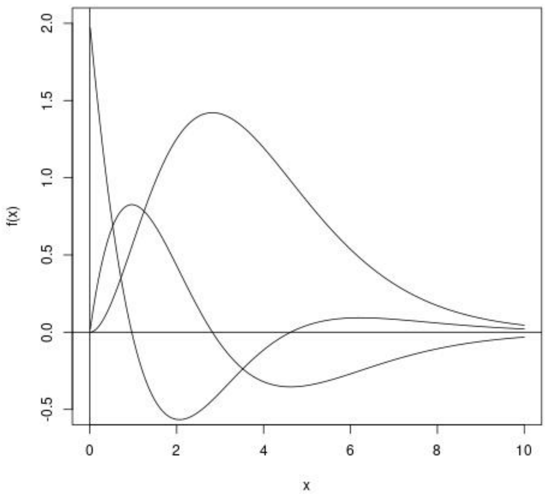

Figure 2: The normalized Planck's Law with respect to frequency and its first and second derivatives.

The normalized function in Equation (6) is shown in Figure 3 along with its first and second derivatives. Unlike the case of the normalization of B_ν, one cannot easily show the normalization of B_λ along with its derivatives in a single graph because the scales of the function and its derivatives are very different.

The advantage of the normalized functions for Planck's Laws is that properties of the normalized functions will apply to all the Planck's Laws. For example, one can find the peak frequency and peak wavelength as well as the inflection points by using a bisection algorithm (Burden and Faires, 1985). The two inflection points of the frequency form of Planck's law are approximately

$$0.342473330825876715941418 1\ \nu_{peak}$$

$$1.638612948477220856894063 5\ \nu_{peak}$$

and the two inflection points of the wavelength form of Planck's law are approximately

$$0.587967455793415628530774 9\ \lambda_{peak}$$

$$1.408887390617991608898521 6\ \lambda_{peak}.$$

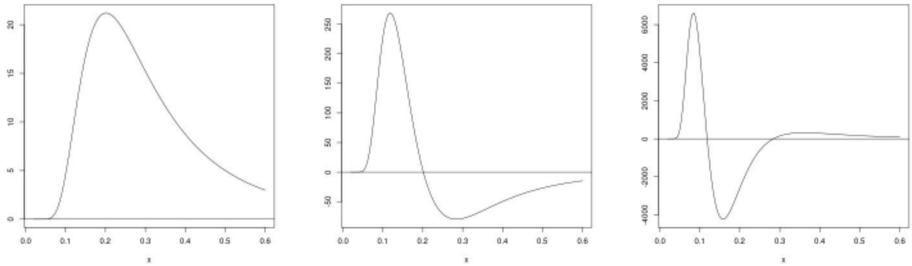

Figure 3: The normalized Planck's Law with respect to wavelength and its first and second derivatives.

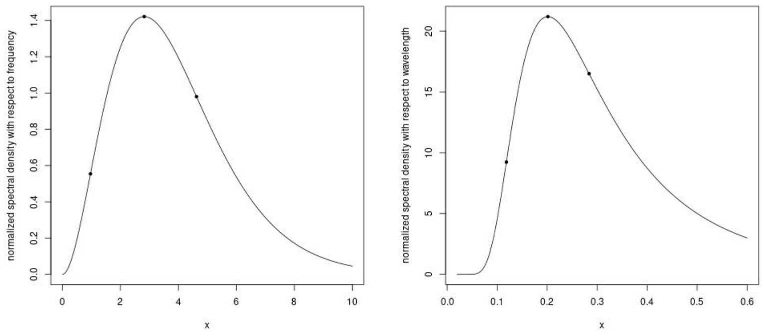

Figure 4: The normalized Planck's Laws with respect to wavelength and frequency with the peaks and inflection points indicated with dots.

The peaks and inflection points are shown using dots on Figure 4. The Sun is approximately a black body with a temperature of about 5790 K. For this temperature, the peak wavelength is $\lambda_{peak} \approx 500$ nm, and the inflection points have wavelengths that are approximately 295 nm and 706 nm.

All of the most commonly used forms of Planck's Law have the same shape as either Equation (5) or Equation (6). The wave number form of Planck's Law has the same shape as the frequency form of Planck's Law, and the angular forms of Planck's Law have the same shape as the corresponding ordinary forms.

Glossary

c = 299792458 m/s is the speed of light in a vacuum (exact).
e = 2.71828182845904523536... is Euler's number.
h = $6.62607015 \times 10^{-34}$ J·Hz^{-1} is Planck's constant (exact).
k = 1.380649×10^{-23} J·K^{-1} is the Boltzmann constant (exact).

References

Burden, Richard L. and Faires, J. Douglas (1985), "2.1 The Bisection Algorithm," *Numerical Analysis* (3rd ed.), PWS Publishers, ISBN 0-87150-857-5

Kendall, D.G. (1984). "Shape Manifolds, Procrustean Metrics, and Complex Projective Spaces," sl Bulletin of the London Mathematical Society. 16(2): 81–121. doi:10.1112/blms/16.2.81.

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